

# ON THE COMPLETE INTEGRABILITY OF THE OSTROVSKY-VAKHENKO EQUATION

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**ABSTRACT.** The complete integrability of the Ostrovsky-Vakhnenko equation is studied by means of symplectic gradient-holonomic and differential-algebraic tools. A compatible pair of polynomial Poissonian structures, Lax type representation and related infinite hierarchies of conservation laws are constructed.

## 1. INTRODUCTION

In 1998 V.O. Vakhnenko investigated high-frequency perturbations in a relaxing barotropic medium. He discovered that this phenomenon is described by a new nonlinear evolution equation. Later it was proved that this equation is equivalent to the reduced Ostrovsky equation [1], which describes long internal waves in a rotating ocean. The nonlinear integro-differential Ostrovsky-Vakhnenko equation

$$(1.1) \quad u_t = -uu_x - D_x^{-1}u$$

on the real axis  $\mathbb{R}$  for a smooth function  $u \in C^{(\infty)}(\mathbb{R}^2; \mathbb{R})$ , where  $D_x^{-1}$  is the inverse-differential operator to  $D_x := \partial/\partial x$ , can be derived [2] as a special case of the Whitham type equation

$$(1.2) \quad u_t = -uu_x + \int_{\mathbb{R}} K(x, y) u_y dy.$$

Here the generalized kernel  $K(x, y) := \frac{1}{2}|x - y|$ ,  $x, y \in \mathbb{R}$  and  $t \in \mathbb{R}$  is an evolution parameter. Different analytical properties of equation (1.1) were analyzed in articles [1, 2, 3], the corresponding Lax type integrability was stated in [5].

Recently by J.C. Brunelli and S. Sakovich in [4] there was demonstrated that Ostrovsky-Vakhnenko equation is a suitable reduction of the well known Camassa-Holm equation that made it possible to construct the corresponding compatible Poisson structures for (1.1), but in a complicated enough non-polynomial form.

In the present work we will reanalyze the integrability of equation (1.1) both from the gradient-holonomic [7, 12, 13], symplectic and formal differential-algebraic points of view. As a result, we will re-derive the Lax type representation for the Ostrovsky-Vakhnenko equation (1.1), construct the related simple enough compatible polynomial Poisson structures and an infinite hierarchy of conservation laws.

## 2. GRADIENT-HOLONOMIC INTEGRABILITY ANALYSIS

Consider the nonlinear Ostrovsky-Vakhnenko equation (1.1) as a suitable nonlinear dynamical system

$$(2.1) \quad du/dt = -uu_x - D_x^{-1}u := K[u]$$

on the smooth  $2\pi$ -periodic functional manifold

$$(2.2) \quad M := \{u \in C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}) : \int_0^{2\pi} u dx = 0\},$$

where  $K : M \rightarrow T(M)$  is the corresponding well-defined smooth vector field on  $M$ .

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*Date:* present.

*1991 Mathematics Subject Classification.* Primary 58A30, 56B05 Secondary 34B15 .

*Key words and phrases.* Lax type integrability, Ostrovsky-Vakhnenko equation, symplectic method, differential-algebraic approach,

We, first, will state that the dynamical system (2.1) on manifold  $M$  possesses an infinite hierarchy of conservation laws, that can signify as a necessary condition for its integrability. For this we need to construct a solution to the Lax gradient equation

$$(2.3) \quad \varphi_t + K'^{*} \varphi = 0,$$

in the special asymptotic form

$$(2.4) \quad \varphi = \exp[-\lambda t + D_x^{-1} \sigma(x; \lambda)],$$

where, by definition, a linear operator  $K'^{*} : T^*(M) \rightarrow T^*(M)$  is, adjoint with respect to the standard convolution  $(\cdot, \cdot)$  on  $T^*(M) \times T(M)$ , the Frechet-derivative of a nonlinear mapping  $K :$

$M \rightarrow T(M) :$

$$(2.5) \quad K'^{*} = u D_x + D_x^{-1}$$

and, respectively,

$$(2.6) \quad \sigma(x; \lambda) \simeq \sum_{j \in \mathbb{Z}_+} \sigma_j[u] \lambda^{-j},$$

as  $|\lambda| \rightarrow \infty$  with some "local" functionals  $\sigma_j : M \rightarrow C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R})$  on  $M$  for all  $j \in \mathbb{Z}_+$ .

By substituting (2.4) into (2.3) one easily obtains the following recurrent sequence of functional relationships:

$$(2.7) \quad \sigma_{j,t} + \sum_{k \leq j} \sigma_{j-k}(u \sigma_k + D_x^{-1} \sigma_{k,t}) - \sigma_{j+1} + (u \sigma_j)_x + \delta_{j,0} = 0$$

for all  $j+1 \in \mathbb{Z}_+$  modulo the equation (2.1). By means of standard calculations one obtains that this recurrent sequence is solvable and

$$(2.8) \quad \begin{aligned} \sigma_0[u] &= 0, \sigma_1[u] = 1, \sigma_2[u] = u_x, \\ \sigma_3[u] &= 0, \sigma_4[u] = u_t + 2u u_x, \\ \sigma_5[u] &= 3/2(u^2)_{xt} + u_{tt} + 2/3(u^3)_{xx} - u_x D_x^{-1} u \end{aligned}$$

and so on. It is easy check that all of functionals

$$(2.9) \quad \gamma_j := \int_0^{2\pi} \sigma_j[u] dx$$

are on the manifold  $M$  conservation laws, that is  $d\gamma_j/dt = 0$  for  $j \in \mathbb{Z}_+$  with respect to the dynamical system (2.1). For instance, at  $j = 5$  one obtains:

$$(2.10) \quad \begin{aligned} \gamma_5 &: = \int_0^{2\pi} \sigma_5[u] dx = \int_0^{2\pi} [3/2(u^2)_{xt} + u_{tt} + 2/3(u^3)_{xx} - u_x D_x^{-1} u] dx = \\ &= \frac{d^2}{dt^2} \int_0^{2\pi} u_{tt} dx - \int_0^{2\pi} u_x D_x^{-1} u dx = \frac{d^2}{dt^2} \int_0^{2\pi} u dx - u D_x^{-1} u \Big|_0^{2\pi} + \int_0^{2\pi} u^2 dx = \\ &= \int_0^{2\pi} u^2 dx, \end{aligned}$$

and

$$(2.11) \quad \begin{aligned} d\gamma_5/dt &= 2 \int_0^{2\pi} u u_t dx = -2 \int_0^{2\pi} u(u u_x + D_x^{-1} u) dx = \\ &= -2 \int_0^{2\pi} u D_x^{-1} u dx = - \int_0^{2\pi} [(D_x^{-1} u)^2]_x dx = (D_x^{-1} u)^2 \Big|_0^{2\pi} = 0, \end{aligned}$$

since owing to the constraint (2.2) the integrals  $(D_x^{-1} u) \Big|_0^{2\pi} = 0$ .

The result stated above allows us to suggest that the dynamical system (2.1) on the functional manifold  $M$  is an integrable Hamiltonian system.

First, we will show that this dynamical system is a Hamiltonian flow

$$(2.12) \quad du/dt = -\vartheta \text{ grad } H[u]$$

with respect to some Poisson structure  $\vartheta : T^*(M) \rightarrow T(M)$  and a Hamiltonian function  $H \in \mathcal{D}(M)$ . Based on the standard symplectic techniques [10, 7, 6, 12] consider the conservation law (2.10) and present it in the scalar "momentum" form:

$$(2.13) \quad -1/2\gamma_5 = \frac{1}{2} \int_0^{2\pi} u_x D_x^{-1} u dx = (1/2 D_x^{-1} u, u_x) := (\psi, u_x)$$

with the co-vector  $\psi := 1/2 D_x^{-1} u \in T^*(M)$  and calculate the corresponding co-Poissonian structure

$$(2.14) \quad \vartheta^{-1} := \psi' - \psi'^{*} = D_x^{-1},$$

or the Poissonian structure

$$(2.15) \quad \vartheta = D_x.$$

The obtained operator  $\vartheta = D_x : T^*(M) \rightarrow T(M)$  is really Poissonian for (2.1) since the following determining symplectic condition

$$(2.16) \quad \psi_t + K'^{*} \psi = \text{grad } \mathcal{L}$$

holds for the Lagrangian function

$$(2.17) \quad \mathcal{L} = \frac{1}{12} \int_0^{2\pi} u^3 dx.$$

As a result of (2.16) one obtains easily that

$$(2.18) \quad du/dt = -\vartheta \text{grad } H[u],$$

where the Hamiltonian function

$$(2.19) \quad H = (\psi, K) - \mathcal{L} = \frac{1}{2} \int_0^{2\pi} [u^3/3 - (D_x^{-1} u)^2/2] dx$$

is an additional conservation law of the dynamical system (2.1). Thus, one can formulate the following proposition.

**Proposition 2.1.** *The Ostrovsky-Vakhnenko dynamical system (2.1) possesses an infinite hierarchy of nonlocal, in general, conservation laws (2.9) and is a Hamiltonian flow (2.18) on the manifold  $M$  with respect to the Poissonian structure (2.15).*

*Remark 2.2.* It is useful to remark here that the existence of an infinite ordered by  $\lambda$ -powers hierarchy of conservations laws (2.9) is a typical property [10, 6, 7, 12] of the Lax type integrable Hamiltonian systems, which are simultaneously bi-Hamiltonian flows with respect to corresponding two compatible Poissonian structures.

As is well known [10, 6, 7, 12], the second Poissonian structure  $\eta : T^*(M) \rightarrow T(M)$  on the manifold  $M$  for (2.1), if it exists, can be calculated as

$$(2.20) \quad \eta^{-1} := \tilde{\psi}' - \tilde{\psi}'^*,$$

where a-covector  $\tilde{\psi} \in T^*(M)$  is a second solution to the determining equation (2.16):

$$(2.21) \quad \tilde{\psi}_t + K'^{*} \tilde{\psi} = \text{grad } \tilde{\mathcal{L}}$$

for some Lagrangian functional  $\tilde{\mathcal{L}} \in \mathcal{D}(M)$ . It can be certainly done by means of simple enough but cumbersome analytical calculations based, for example, on the asymptotical small parameter method [7, 12, 13] and on which we will not stop here.

Instead of this we will shall apply the direct differential-algebraic approach to dynamical system (2.1) and reveal its Lax type representation both in the differential scalar and in canonical matrix Zakharov-Shabat forms. Moreover, we will construct the naturally related compatible polynomial Poissonian structures for Ostrovsky -Vakhnenko dynamical system (2.1) and generate an infinite hierarchy of commuting to each other nonlocal conservation laws.

### 3. LAX TYPE REPRESENTATION AND COMPATIBLE POISSONIAN STRUCTURES - THE DIFFERENTIAL-ALGEBRAIC APPROACH

We will start with construction of the polynomial differential ring  $\mathcal{K}\{u\} \subset \mathcal{K} := \mathbb{R}\{\{x, t\}\}$  generated by a fixed functional variable  $u \in \mathbb{R}\{\{x, t\}\}$  and invariant with respect to two differentiations  $D_x := \partial/\partial x$  and  $D_t := \partial/\partial t + u\partial/\partial x$ , satisfying the Lie-algebraic commutator relationship

$$(3.1) \quad [D_x, D_t] = u_x D_x.$$

Since the Lax type representation for the dynamical system (2.1) can be interpreted [8, 12] as the existence of a finite-dimensional invariant differential ideal  $\mathcal{I}_N\{u\} \subset \mathcal{K}\{u\}$ , realizing the corresponding finite-dimensional representation of the the Lie-algebraic commutator relationship (3.1), this ideal can be presented as

$$(3.2) \quad \mathcal{I}_N\{u\} := \left\{ \sum_{j=\overline{0, N}} g_j D_x^j f[u] \in \mathcal{K}\{u\} : g_j \in \mathcal{K}, j = \overline{0, N} \right\},$$

where an element  $f[u] \in \mathcal{K}\{u\}$  and  $N \in \mathbb{Z}_+$  are fixed. The  $D_x$ -invariance of ideal (3.2) will be *a priori* evident, if the function  $f[u] \in \mathcal{K}\{u\}$  satisfies the linear differential relationship

$$(3.3) \quad D_x^{N+1} f = \sum_{k=0}^N a_k[u] D_x^k f$$

for some coefficients  $a_j[u] \in \mathcal{K}\{u\}$ ,  $j = \overline{0, N}$ , but its  $D_t$ -invariance strongly depends on the element  $f[u] \in \mathcal{K}\{u\}$ , which can be found from the functional relationship (2.3) on the element  $\varphi[u; \lambda] := \text{grad } \gamma(\lambda) \in \mathcal{K}\{u\}$ ,  $\gamma(\lambda) := \int_0^{2\pi} \sigma(x; \lambda) dx$ , rewritten in the following form:

$$(3.4) \quad D_x D_t \varphi = -\varphi.$$

From the right hand side one follows that there exists an element  $\eta := \eta[u] = -D_t \varphi[u] \in \mathcal{K}\{u\}$ , such that

$$(3.5) \quad \varphi = D_x \eta.$$

Having substituted (3.5) into the left hand side of (3.4) one finds easily that

$$(3.6) \quad \begin{aligned} D_x D_t \eta - u_x \eta_x &= D_x D_t \eta - u_x \varphi = \\ &= D_x D_t \eta - u_x \text{grad } \gamma(\lambda) = \\ &= D_x (D_t \eta - \gamma[u, \lambda]) = \eta, \end{aligned}$$

where we have put, by definition,  $\gamma(\lambda) := \int_0^{2\pi} \gamma[u; \lambda] dx$  for a suitably chosen density element  $\gamma[u; \lambda] \in \mathcal{K}\{u\}$ . As an evident result of (3.6) one derives that there exists an element  $\rho := \rho[u] \in \mathcal{K}\{u\}$ , such that

$$(3.7) \quad \eta = D_x \rho.$$

Turning back to the relationships (3.5) and (3.7) one obtains that the following differential representation

$$(3.8) \quad \varphi = D_x^2 \rho$$

holds.

As a further step, we can try to realize the differential ideal (3.2) by means of the generating element  $f[u] \implies \rho[u] \in \mathcal{K}\{u\}$ , defined by the relationship (3.8). But, as it is easy to check, the obtained this way differential ideal is not finite-dimensional. So, for a future calculating convenience, we will represent the element  $\rho[u] \in \mathcal{K}\{u\}$  in the following natural factorized form:

$$(3.9) \quad \rho := \bar{f} f,$$

where elements  $f, \bar{f} \in \mathcal{K}\{u\}$  satisfy, by definition, the adjoint pairs of the following differential relationships:

$$(3.10) \quad \begin{aligned} D_x^{N+1} f &= \sum_{k=0}^N a_k[u] D_x^k f, \\ (-1)^{N+1} D_x^{N+1} \bar{f} &= \sum_{k=0}^N (-1)^k (D_x^k a_k[u]) \bar{f}, \end{aligned}$$

and

$$(3.11) \quad D_t f = \sum_{j=0}^{N-1} b_j D_x^j f, \quad D_t \bar{f} = -u_x \bar{f} + \sum_{j=0}^{N-1} (-1)^{j+1} (D_x^j b_j) f,$$

for some elements  $b_j \in \mathcal{K}\{u\}$ ,  $j = 0, N-1$ , and check the finite-dimensional  $D_x$ - and  $D_t$ -invariance of the corresponding ideal (3.2), generated by the element  $f \in \mathcal{K}\{u\}$ .

Now it is easy to check by means of simple enough calculations, based on the relationship (3.4) and (3.8), that the following differential equalities

$$(3.12) \quad \begin{aligned} D_x(D_t \varphi) &= -\varphi, & D_x(D_t^2 \varphi) &= -u_x \varphi - D_t \varphi, \\ D_x(D_t^3 \varphi) &= u \varphi - 2u_x D_t \varphi - D_t^2 \varphi, \\ D_x(D_t^4 \varphi) &= -(u u_x + D_x^{-1} u) \varphi + (4u_x^2 + 3u) D_t \varphi - 2u_x D_t^2 \varphi - D_t^3 \varphi, \dots, \end{aligned}$$

and their consequences

$$(3.13) \quad \begin{aligned} D_t D_x^2 \rho &= -\rho_x, & D_x(D_t \rho_x) &= u_x \rho_{xx} - D_x \rho, \\ D_x^2(D_t \rho) &= D_x(u_x D_x \rho - \rho) + u_{xx} D_x^2 \rho, \dots, \end{aligned}$$

hold. Taking into account the independence of the sets of functional elements  $\{f, D_x f, D_x^2 f, \dots, D_x^{N-1} f\} \subset \mathcal{K}\{u\}$  and  $\{\bar{f}, D_x \bar{f}, D_x^2 \bar{f}, \dots, D_x^{N-1} \bar{f}\} \subset \mathcal{K}\{u\}$ , the relationships (3.13) jointly with (3.9), (3.10) and (3.11) make it possible to state the following lemma.

**Lemma 3.1.** *The set (3.2) represents a  $D_x$ - and  $D_t$ -invariant differential ideal in the ring  $\mathcal{K}$  for all  $N \geq 2$ .*

*Proof.* This result easily follows from the fact that for number  $N \geq 2$  all of the relationships (3.13) persist to be compatible upon taking into account the differential expressions (3.9) and (3.11). Contrary to that, at  $N = 1$  they become not compatible.  $\square$

As a corollary of Lemma 3.1, having put in (3.2) and (3.11) the number  $N = 2$ , one finds easily by means of elementary enough calculations that the related differential ideal  $\mathcal{I}_2\{u\}$  lasts to be invariant, if the differential Lax type relationships

$$(3.14) \quad D_x^3 f = -\mu \bar{u} f, \quad D_x^3 \bar{f} = \mu \bar{u} \bar{f},$$

and

$$(3.15) \quad D_t f = \mu^{-1} D_x^2 f + u_x f, \quad D_t \bar{f} = -\mu^{-1} D_x^2 \bar{f} - 2u_x \bar{f},$$

where  $\bar{u} := u_{xx} + 1/3$ ,  $\mu \in \mathbb{C} \setminus \{0\}$  is an arbitrary complex parameter, hold. Moreover, they exactly coincide with those found before in [5]. The obtained above differential relationships (3.14) and (3.15) can be equivalently rewritten in the following matrix Zakharov-Shabat type form:

$$(3.16) \quad D_t h = \hat{q}[u; \mu] h, \quad D_x h = \hat{l}[u; \mu] h,$$

where matrices

$$(3.17) \quad \hat{q}[u; \mu] := \begin{pmatrix} u_x & 0 & 1/\mu \\ -1/3 & 0 & 0 \\ 0 & -1/3 & -u_x \end{pmatrix}, \quad \hat{l}[u; \mu] := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu \bar{u} & 0 & 0 \end{pmatrix}$$

and  $h := (f, D_x f, D_x^2 f)^\top \in \mathcal{K}\{u\}^3$ .

Based further on the obtained differential relationships (3.14) and (3.15), one obtains that the compatibility condition (3.4) gives rise to the following important relationship

$$(3.18) \quad -\vartheta \varphi = D_x^2 D_t \varphi = 3\mu^2 \eta \varphi,$$

where the polynomial integro-differential operator

$$(3.19) \quad \eta := \partial^{-1} \bar{u} \partial^{-3} \bar{u} \partial^{-1} + 4\partial^{-2} \bar{u} \partial^{-1} \bar{u} \partial^{-2} + 2(\partial^{-2} \bar{u} \partial^{-2} \bar{u} \partial^{-1} + \partial^{-1} \bar{u} \partial^{-2} \bar{u} \partial^{-2})$$

is skewsymmetric on the functional manifold  $M$  and presents the second compatible Poisson structure for the Ostrovsky-Vakhnenko dynamical system (2.1).

Based now on the recurrent relationships following from substitution of the asymptotic expansion

$$(3.20) \quad \varphi \simeq \sum_{j \in \mathbb{Z}_+} \varphi_j \xi^{-j}, \quad \xi := -1/(3\mu^2),$$

into (3.18), one can determine a new infinite hierarchy of conservations laws for dynamical system (2.1):

$$(3.21) \quad \tilde{\gamma}_j := \int_0^1 ds(\varphi_j[us], u),$$

for  $j \in \mathbb{Z}_+$ , where

$$(3.22) \quad \varphi_j = \Lambda^j \varphi_0, \quad \vartheta \varphi_0 = 0,$$

and the recursion operator  $\Lambda := \vartheta^{-1} \eta : T^*(M) \rightarrow T^*(M)$  satisfies the standard Lax type representation:

$$(3.23) \quad \Lambda_t = [\Lambda, K'^*].$$

The obtained above results can be formulated as follows.

**Proposition 3.2.** *The Ostrovsky-Vakhnenko dynamical system (2.1) allows the standard differential Lax type representation (3.14), (3.15) and defines on the functional manifold  $M$  an integrable bi-Hamiltonian flow with compatible Poisson structures (2.15) and (3.19). In particular, this dynamical system possesses an infinite hierarchy of nonlocal conservation laws (3.21), defined by the gradient elements (3.22).*

*Remark 3.3.* It is useful to remark here that the existence of an infinite  $\lambda$ -powers ordered hierarchy of conservations laws (2.9) is a typical property [10, 6, 7, 12] of the Lax type integrable Hamiltonian systems, which are simultaneously bi-Hamiltonian flows with respect to corresponding compatible Poissonian structures.

*Remark 3.4.* It is interesting to observe that our second polynomial Poisson structure (3.19) differs from that obtained recently in [4], which contains the rational power factors.

It is easy to construct making use of the differential expressions (3.14) and (3.15) a slightly different from (3.16) matrix Lax type representation of the Zakharov-Shabat form for the dynamical system (1.1).

Really, if to define the "spectral" parameter  $\mu := 1/(9\lambda) \in \mathbb{C} \setminus \{0\}$  and new basis elements of the invariant differential ideal (3.2):

$$(3.24) \quad g_1 := -3D_x f, \quad g_2 := f, \quad g_3 := 9\lambda D_x^2 f + u_x f,$$

then relationships (3.14) and (3.15) can be rewritten as follows:

$$(3.25) \quad D_t g = q[u; \lambda]g, \quad D_x g = l[u; \lambda]g,$$

where matrices

$$(3.26) \quad q[u; \lambda] := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -u & 0 \end{pmatrix}, \quad l[u; \lambda] := \begin{pmatrix} 0 & u_x/(3\lambda) & -1/(3\lambda) \\ -1/3 & 0 & 0 \\ -u_x/3 & -1/3 & 0 \end{pmatrix}$$

coincide with those of [5, 4] and satisfy the following Zakharov-Shabat type compatibility condition:

$$(3.27) \quad D_t l = [q, l] + D_x q - l D_x u.$$

*Remark 3.5.* As it was already mentioned above, the Lax type representation (3.26) of the Ostrovsky-Vakhnenko dynamical system (1.1) was obtained in [5] by means of a suitable limiting reduction of the Degasperis-Processi equation

$$(3.28) \quad u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0.$$

For convenience, let us rewrite the latter in the following form:

$$(3.29) \quad D_t z = -3z D_x u, \quad z = u - D_x^2 u,$$

where differentiations  $D_t := \partial/\partial t + u\partial/\partial x$  and  $D_x := \partial/\partial x$  satisfy the Lie- algebraic relationship (3.1). It appears to be very impressive that equation (3.28) is itself a special reduction of a new Lax type integrable Riemann type hydrodynamic system, proposed and studied (at  $s = 2$ ) recently in [11]:

$$(3.30) \quad D_t^{N-1} u = \bar{z}_x^s, \quad D_t \bar{z} = 0,$$

where  $s, N \in \mathbb{N}$  are arbitrary natural numbers. Really, having put, by definition,  $z := \bar{z}_x^s$  and  $s = 3$ , from (3.30) one easily obtains the following dynamical system:

$$(3.31) \quad \begin{aligned} D_t^{N-1} u &= z, \\ D_t z &= -3z D_x u, \end{aligned}$$

coinciding with the Degasperis-Processi equation (3.29) if to make the identification  $z = u - D_x^2 u$ . As a result, we have stated that a function  $u \in C^\infty(\mathbb{R}^2; \mathbb{R})$ , satisfying for an arbitrary  $N \in \mathbb{N}$  the generalized Riemann type hydrodynamical equation

$$(3.32) \quad D_t^{N-1} u = u - D_x^2 u,$$

simultaneously solves the Degasperis-Processi equation (3.28). In particular, having put  $N = 2$ , we obtain that solutions to the Burgers type equation

$$(3.33) \quad D_t u = u - D_x^2 u$$

are solving also the Degasperis-Processi equation (3.28). It means, in particular, that the reduction procedure of the work [5] can be also applied to the Lax type integrable Riemann type hydrodynamic system (3.30), giving rise to a related Lax type representation for the Ostrovsky-Vakhnenko dynamical system (1.1).

#### 4. CONCLUSION

We have showed that the Ostrovsky-Vakhnenko dynamical system is naturally embedded into the general Lax type integrability scheme [10, 6, 7, 12], whose main ingredients such as the corresponding compatible Poissonian structures and Lax type representation can be effectively enough retrieved by means of direct modern integrability tools, such as the differential-geometric, differential-algebraic and symplectic gradient holonomic approaches. We have also demonstrated the relationship of the Ostrovsky-Vakhnenko equation 1.1 with a generalize Riemann type hydrodynamic system, studied recently in [11] and its reduction.

#### 5. ACKNOWLEDGEMENTS

Author acknowledges the Scientific and Technological Research Council of Turkey (TUBITAK/NASU-111T558 Project) for a partial support of his research.

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